Weak*-sequential properties of Johnson-Lindenstrauss spaces

Banach spaces and their Applications

International conference dedicated to the 70th anniversary of Professor A.M. Plichko

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- $K$ is said to be **sequential** if any sequentially closed subspace is closed.
- $K$ is said to have **countable tightness** if for every subspace $F$ of $K$, every point in the closure of $F$ is in the closure of a countable subspace of $F$. 
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\[ \Downarrow \]

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A Banach space with weak*-FU dual ball is said to have **weak*-angelic** dual.
$X$ has weak*-angelic dual
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\( \Downarrow \)

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\[\implies\]

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Definition

Let $X$ be a Banach space.

- $X$ is said to have **property (E)** (of Efremov) if every point in the weak*-closure of any convex subset $C \subset B_{X^*}$ is the weak*-limit of a sequence in $C$.

- $X$ is said to have **property (E)'** if every weak*-sequentially closed convex set in the dual ball is weak*-closed.

- $X$ has **property (C)** of Corson if and only if every point in the closure of $C$ is in the weak*-closure of a countable subset of $C$ for every convex set $C$ in $B_{X^*}$ (Pol’s characterization).
Definition

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$\downarrow$

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$(B_{X^*}, w^*)$ has countable tightness
$X$ has weak*-angelic dual $\implies X$ has property ($\mathcal{E}$)

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$X$ has weak*-sequentially compact dual ball $\implies (B_{X^*}, w^*)$ has countable tightness
\(X\) has weak*-angelic dual \(\Rightarrow X\) has property \((\mathcal{E})\)

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(consistently)
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Question (A. Plichko, 2014)

Does every Banach space with weak*-sequential dual ball have weak*-angelic dual?
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Open Problem

Is having weak*-sequential dual ball a three-space property?
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Open Problem
Is having weak*-sequential dual ball a three-space property?
Is property $(\mathcal{E}')$ a three-space property?
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Let $\mathcal{F} = \{N_r : r \in \Gamma\}$ be a maximal almost disjoint (MAD) family in $\mathbb{N}$. The Johnson-Lindenstrauss space $\mathcal{JL}_2$ is defined as the completion of $\text{span} (c_0 \cup \{\chi_{N_r} : r \in \Gamma\}) \subset \ell_\infty$ with respect to the norm:

$$\left\| x + \sum_{1 \leq i \leq k} a_i \chi_{N_{r_i}} \right\| = \max \left\{ \left\| x + \sum_{1 \leq i \leq k} a_i \chi_{N_{r_i}} \right\|_\infty, \left( \sum_{1 \leq i \leq k} |a_i|^2 \right)^{\frac{1}{2}} \right\}.$$
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\| x + \sum_{1 \leq i \leq k} a_i \chi_{N_{r_i}} \| = \max \left\{ \| x + \sum_{1 \leq i \leq k} a_i \chi_{N_{r_i}} \|_\infty , \left( \sum_{1 \leq i \leq k} |a_i|^2 \right)^{\frac{1}{2}} \right\}.
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If we just consider the supremum norm in the definition then we obtain the space $\mathcal{JL}_0$. 

Let $\mathcal{F} = \{N_r : r \in \Gamma\}$ be a maximal almost disjoint (MAD) family in $\mathbb{N}$. The Johnson-Lindenstrauss space $JL_2$ is defined as the completion of $\text{span}(c_0 \cup \{\chi_{N_r} : r \in \Gamma\}) \subset \ell_\infty$ with respect to the norm:

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4. If $S_0 = \{ e_n \}_{n \in \mathbb{N}}$ is the canonical basis of $\ell_1$ in $JL_2^*$, then the sequential closure of $S_0$ is $S_1 = S_0 \cup \{ e_\alpha \}_{\alpha \in \Gamma}$.
Let $\mathcal{F} = \{N_r : r \in \Gamma\}$ be a maximal almost disjoint (MAD) family in $\mathbb{N}$. The Johnson-Lindenstrauss space $JL_2$ is defined as the completion of span $(c_0 \cup \{\chi_{N_r} : r \in \Gamma\}) \subset \ell_\infty$ with respect to the norm:

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In particular, $JL_2$ does not have weak*-angelic dual.
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Question (Plichko and Yost, 1980)

Does property (C) of Corson imply property (E)?

Does Johnson-Lindestrauss space JL have property (E)?

J. Moore and C. Brech provided consistent examples of Banach spaces with property (C) but without property (E). Indeed, these examples do not have property (E′).

Theorem (A. Avilés, G.M.C., J. Rodríguez, 2019)

Under CH, there exists a MAD family F− for which the corresponding Johnson-Lindestrauss JL does not have property (E).

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Under CH, there exists a MAD family $\mathcal{F}^+$ for which the corresponding Johnson-Lindestrauss $JL_2$ has property (E).
Comments about the construction of $\mathcal{F}^+$
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$JL_2(\mathcal{F}^+)$ has the following property: every bounded sequence $(x_n^*)_{n \in \mathbb{N}}$ for which 0 is a weak*-cluster point has a subsequence $(x_{n_k}^*)_{k \in \mathbb{N}}$ such that

$$\lim_{k} \frac{1}{k}(x_{n_1}^* + \ldots + x_{n_k}^*) = 0.$$
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This condition guarantees that $\mathcal{J}\mathcal{L}_2(\mathcal{F}^+)$ has property ($\mathcal{E}$). The family $\mathcal{F}^+$ is the union of an increasing sequence of countable almost disjoint families $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$ which is constructed by induction.
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1. $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{N\}$ is an almost disjoint family;
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$\mathcal{J}\mathcal{L}_2(\mathcal{F}^+)$ has the following property: every bounded sequence $(x^*_n)_{n \in \mathbb{N}}$ for which 0 is a weak*-cluster point has a subsequence $(x^*_{n_k})_{k \in \mathbb{N}}$ such that

$$\lim_k \frac{1}{k}(x^*_1 + \ldots + x^*_k) = 0.$$  

This condition guarantees that $\mathcal{J}\mathcal{L}_2(\mathcal{F}^+)$ has property ($\mathcal{E}$). The family $\mathcal{F}^+$ is the union of an increasing sequence of countable almost disjoint families $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$ which is constructed by induction. At step $\alpha$ we have a countable almost disjoint family $\mathcal{F}_\alpha$, a countable family $\mathcal{S}_\alpha$ of sequences whose arithmetic-means converge to zero in $\mathcal{J}\mathcal{L}_2(\mathcal{F}_\alpha)^*$ and a sequence $(x^*_n)_{n \in \mathbb{N}}$ having zero as a weak*-cluster point in $\mathcal{J}\mathcal{L}_2(\mathcal{F}_\alpha)^*$. Passing to a suitable subsequence $(x^*_{n_k})_{k \in \mathbb{N}}$ we can find an infinite set $N \subseteq \mathbb{N}$ such that:

1. $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{ N \}$ is an almost disjoint family;
2. $\lim_k \frac{1}{k}(x^*_1 + \ldots + x^*_{n_k})(\chi_N) = 0$;
3. $\lim_k \frac{1}{k}(y^*_1 + \ldots + y^*_k)(\chi_N) = 0$ for every sequence $(y^*_k)_{k \in \mathbb{N}}$ in $\mathcal{S}_\alpha$.
Comments about the construction of $\mathcal{F}^+$

$\mathcal{J}\mathcal{L}_2(\mathcal{F}^+)$ has the following property: every bounded sequence $(x^*_n)_{n \in \mathbb{N}}$ for which 0 is a weak*-cluster point has a subsequence $(x^*_{n_k})_{k \in \mathbb{N}}$ such that
\[
\lim_k \frac{1}{k} (x^*_n + \ldots + x^*_n) = 0.
\]
This condition guarantees that $\mathcal{J}\mathcal{L}_2(\mathcal{F}^+)$ has property $(\mathcal{E})$. The family $\mathcal{F}^+$ is the union of an increasing sequence of countable almost disjoint families $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$ which is constructed by induction. At step $\alpha$ we have a countable almost disjoint family $\mathcal{F}_\alpha$, a countable family $\mathcal{S}_\alpha$ of sequences whose arithmetic-means converge to zero in $\mathcal{J}\mathcal{L}_2(\mathcal{F}_\alpha)^*$ and a sequence $(x^*_n)_{n \in \mathbb{N}}$ having zero as a weak*-cluster point in $\mathcal{J}\mathcal{L}_2(\mathcal{F}_\alpha)^*$. Passing to a suitable subsequence $(x^*_{n_k})_{k \in \mathbb{N}}$ we can find an infinite set $\mathcal{N} \subseteq \mathbb{N}$ such that:

1. $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{\mathcal{N}\}$ is an almost disjoint family;
2. $\lim_k \frac{1}{k} (x^*_n + \ldots + x^*_n)(\chi_{\mathcal{N}}) = 0$;
3. $\lim_k \frac{1}{k} (y^*_1 + \ldots + y^*_k)(\chi_{\mathcal{N}}) = 0$ for every sequence $(y^*_k)_{k \in \mathbb{N}}$ in $\mathcal{S}_\alpha$.

Then take $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \cup \{(x^*_{n_k})_{k \in \mathbb{N}}\}$ and $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{\mathcal{N}\}$. Under CH, every sequence having zero as a weak*-cluster point is considered at some step $\alpha < \omega_1$ and therefore $\mathcal{J}\mathcal{L}_2(\mathcal{F}^+)$ has property $(\mathcal{E})$. 

Comments about the construction of $\mathcal{F}^+$

$\mathcal{JL}_2(\mathcal{F}^+)$ has the following property: every bounded sequence $(x^*_n)_{n \in \mathbb{N}}$ for which 0 is a weak*-cluster point has a subsequence $(x^*_{n_k})_{k \in \mathbb{N}}$ such that
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This condition guarantees that $\mathcal{JL}_2(\mathcal{F}^+)$ has property (E). The family $\mathcal{F}^+$ is the union of an increasing sequence of countable almost disjoint families $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$ which is constructed by induction. At step $\alpha$ we have a countable almost disjoint family $\mathcal{F}_\alpha$, a countable family $\mathcal{S}_\alpha$ of sequences whose arithmetic-means converge to zero in $\mathcal{JL}_2(\mathcal{F}_\alpha)^*$ and a sequence $(x^*_n)_{n \in \mathbb{N}}$ having zero as a weak*-cluster point in $\mathcal{JL}_2(\mathcal{F}_\alpha)^*$. Passing to a suitable subsequence $(x^*_{n_k})_{k \in \mathbb{N}}$ we can find an infinite set $N \subseteq \mathbb{N}$ such that:

1. $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{N\}$ is an almost disjoint family;
2. $\lim_k \frac{1}{k} (x^*_{n_1} + \ldots + x^*_{n_k})(\chi_N) = 0$;
3. $\lim_k \frac{1}{k} (y^*_1 + \ldots + y^*_k)(\chi_N) = 0$ for every sequence $(y^*_k)_{k \in \mathbb{N}}$ in $\mathcal{S}_\alpha$.

Then take $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \cup \{(x^*_{n_k})_{k \in \mathbb{N}}\}$ and $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{N\}$. Under CH, every sequence having zero as a weak*-cluster point is considered at some step $\alpha < \omega_1$ and therefore $\mathcal{JL}_2(\mathcal{F}^+)$ has property (E).
Comments about the construction of $\mathcal{F}^-$

Recall that $\mathcal{J}L_2(\mathcal{F})^* = \ell_1 \oplus \ell_2(\mathcal{F})$. 
Comments about the construction of $\mathcal{F}^-$

Recall that $JL_2(\mathcal{F})^* = \ell_1 \oplus \ell_2(\mathcal{F})$. $\mathcal{F}^-$ is constructed in such a way that $\text{co}([e_n^*: n \in \mathbb{N}])$ does not contain weak*-null sequences in $JL_2(\mathcal{F}^-)$. 
Comments about the construction of $\mathcal{F}^-$

Recall that $J\mathcal{L}_2(\mathcal{F})^* = \ell_1 \oplus \ell_2(\mathcal{F})$. $\mathcal{F}^-$ is constructed in such a way that $\text{co}(\{e_n^*: n \in \mathbb{N}\})$ does not contain weak*-null sequences in $J\mathcal{L}_2(\mathcal{F}^-)$. $\mathcal{F}^-$ is again obtained as the union of an increasing sequence of countable almost disjoint families $(\mathcal{F}_\alpha)_{\alpha<\omega_1}$. 
Comments about the construction of $\mathcal{F}^-$

Recall that $\mathcal{J}\mathcal{L}_2(\mathcal{F})^* = \ell_1 \oplus \ell_2(\mathcal{F})$. $\mathcal{F}^-$ is constructed in such a way that $\text{co}(\{ e_n^* : n \in \mathbb{N} \})$ does not contain weak*-null sequences in $\mathcal{J}\mathcal{L}_2(\mathcal{F}^-)$.

$\mathcal{F}^-$ is again obtained as the union of an increasing sequence of countable almost disjoint families $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$.

At each step $\alpha$ we consider a weak*-null sequence $(x_n^*)_{n \in \mathbb{N}} \in \text{co}(\{ e_n^* : n \in \mathbb{N} \}) \subset \mathcal{J}\mathcal{L}_2(\mathcal{F}_\alpha)^*$ and we kill it by finding a set $N \in \mathbb{N}$ such that

1. $\limsup_k x_n^*(\chi_N) > 0$;
Comments about the construction of $\mathcal{F}^-$

Recall that $JL_2(\mathcal{F})^* = \ell_1 \oplus \ell_2(\mathcal{F})$. $\mathcal{F}^-$ is constructed in such a way that $\text{co}(\{e_n^* : n \in \mathbb{N}\})$ does not contain weak*-null sequences in $JL_2(\mathcal{F}^-)$. $\mathcal{F}^-$ is again obtained as the union of an increasing sequence of countable almost disjoint families $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$.

At each step $\alpha$ we consider a weak*-null sequence $(x_n^*)_{n \in \mathbb{N}} \in \text{co}(\{e_n^* : n \in \mathbb{N}\}) \subset JL_2(\mathcal{F}_\alpha)^*$ and we kill it by finding a set $N \in \mathbb{N}$ such that

1. $\limsup_k x_n^*(\chi_N) > 0$;
2. $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{N\}$ is an almost disjoint family.
Comments about the construction of $F^-$

Recall that $JL_2(F)^* = \ell_1 \oplus \ell_2(F)$. $F^-$ is constructed in such a way that $\text{co}(\{e_n^* : n \in \mathbb{N}\})$ does not contain weak*-null sequences in $JL_2(F^-)$. $F^-$ is again obtained as the union of an increasing sequence of countable almost disjoint families $(F_\alpha)_{\alpha < \omega_1}$.

At each step $\alpha$ we consider a weak*-null sequence $(x_n^*)_{n \in \mathbb{N}} \in \text{co}(\{e_n^* : n \in \mathbb{N}\}) \subset JL_2(F_\alpha)^*$ and we kill it by finding a set $N \in \mathbb{N}$ such that

1. $\limsup_k x_n^*(\chi_N) > 0$;
2. $F_{\alpha+1} = F_\alpha \cup \{N\}$ is an almost disjoint family.

Under CH, after $\omega_1$ steps no weak*-null sequence in $\text{co}(\{e_n^* : n \in \mathbb{N}\})$ survives.
$X$ has weak*-angelic dual

$X$ has weak*-sequential dual ball

$X$ has weak*-sequentially compact dual ball

$(B_{X^*}, w^*)$ has countable tightness

$X$ has weak*-convex block compact dual ball

$X$ has property (ε)

$X$ has property (ε′)

$X$ has property (C)

(consistently)
$X$ has weak*-angelic dual

$X$ has weak*-sequential dual ball

$X$ has property ($\mathcal{E}$)

$X$ has property ($\mathcal{E}'$)

$X$ has weak*-convex block compact dual ball

$(B_{X^*}, w^*)$ has countable tighness

$X$ has property ($C$)
$X$ has weak*-angelic dual $\iff X$ has property $(\mathcal{E})$

$X$ has weak*-sequential dual ball $\iff X$ has property $(\mathcal{E}')$

$X$ has weak*-sequentially compact dual ball $\iff X$ has property $(\mathcal{C})$

$(B_{X^*}, w^*)$ has countable tightness $\iff X$ has property $(\mathcal{C})$
\( X \) has weak*-angelic dual

\( X \) has weak*-sequential dual ball

\( X \) has weak*-sequentially compact dual ball

\( (B_{X^*}, w^*) \) has countable tightness

\( X \) has weak*-convex block compact dual ball

\( X \) has property \((\mathcal{E})\)

\( X \) has property \((\mathcal{E}')\)

\( X \) has property \((C)\)

\((\text{consistently})\)

\((\text{consistently})\)

\((\text{consistently})\)
$X$ has weak*-angelic dual

$X$ has weak*-sequential dual ball

$X$ has weak*-sequentially compact dual ball

$(B_{X^*}, w^*)$ has countable tightness

$X$ has property $(\mathcal{E})$

$X$ has property $(\mathcal{E}')$

$X$ has weak*-convex block compact dual ball

$X$ has property $(C)$

(consistently) $\checkmark$
Open Problem

Can $\mathcal{F}^+$ or $\mathcal{F}^-$ be constructed in ZFC without any extra set-theoretic assumption?
A. Avilés, G. Martínez-Cervantes, J. Rodríguez
Weak*-sequential properties of Johnson-Lindenstrauss spaces.

W.B. Johnson, J. Lindenstrauss,
Some remarks on weakly compactly generated Banach spaces.

G. Martínez-Cervantes,
Banach spaces with weak*-sequential dual ball.

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Three sequential properties of dual Banach spaces in the weak* topology.

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Complemented and uncomplemented subspaces of Banach spaces.