Mackey-Arens Theorem for locally convex spaces.

Topological groups.

Vilenkin’s Contribution: Locally quasi-convex groups (lqc-groups).

Duality for groups and the failure of Mackey-Arens Theorem for lqc-groups.

g-barrelled groups: a satisfactory class of groups.

The talk is mainly based upon the paper:

*On Mackey topology for groups*

by M. J. Chasco, E. Martín-Peinador and V. Tarieladze

Vector space dualities

Let

- $E$ vector space over $\mathbb{R}$.
- $E^a = \{ f : E \rightarrow \mathbb{R} \text{ linear mapping} \}$ is a vector space called the algebraic dual of $E$.
- $F$ subspace of $E^a$
- Duality $(E, F)$ is the pair. If $F$ separates points of $E$, it is a separated duality
- $(E, \tau)$ be a topological vector space (tvs).

The dual of $(E, \tau)$ is:

$$(E, \tau)^* := CLin(E, \mathbb{R})$$

If $(E, \tau)$ has a basis of convex nbhds of 0, it is a locally convex space (lcs).
Topologies associated to a duality \((E, F)\)

- \(\sigma(E, F)\) is the weak topology on \(E\), corresponding to the family \(F\). It is a linear Hausdorff topology provided \(F\) separates points of \(E\).
- \(\sigma(F, E)\) is the topology on \(F\) of pointwise convergence on the elements of \(E\).
- If \((E, \tau)\) is a tvs, and \(E^*\) its dual there is a **natural** duality: \((E, E^*)\).

**Topology compatible with a vector duality**

1) A linear topology \(\nu\) on a vector space \(E\) is compatible with the duality \((E, F)\) if \((E, \nu)^* = F\).

2) For a tvs \((E, \tau)\), a linear topology \(\nu\) is called compatible with \(\tau\), if it is compatible with the natural duality \((E, E^*)\).

\(\sigma(E, F)\) y \(\sigma(F, E)\) are locally convex topologies compatible with the duality \((E, F)\), on \(E\) and \(F\) respectively.
Let \((E, \tau)\) be a tvs.

- Denote by \(C(E_\tau)\) the family of all the locally convex topologies on \(E\) compatible with \(\tau\).
- \(C(E_\tau)\) is partially ordered by \(\subseteq\).
- \(\sigma(E, E^*)\) is the bottom element in \(C(E_\tau)\)

**Mackey-Arens Theorem**

**i)** There exists a top element \(\mu\) in \(C(E_\tau)\). It is called the Mackey topology for \((E, \tau)\).

**ii)** The Mackey topology in \(C(E_\tau)\) is the topology of uniform convergence on the family \(\mathcal{G}\) of all the \(\sigma(E^*, E)\)-compact and absolutely convex subsets of \(E^*\).
Topological groups (tg)

Elementary examples of tg

- The group of integers $\mathbb{Z}$ endowed with the discrete topology.
- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ with the respective euclidean (or induced) topologies.
- $\mathbb{T}$ the multiplicative group of complex number modulus 1.
- The linear group $\text{GL}(\mathbb{R}, n)$ with the topology induced by that of $\mathbb{R}^{n^2}$. (Non abelian)
- Any topological vector space with respect to addition.
- The group of integers endowed with the p-adic topology $\tau_p$. A zero nbd basis for $\tau_p$ (p is a prime number) is given by the family $\mathcal{U} := \{ p^n\mathbb{Z} : n \in \mathbb{N} \}$.
- $\tau_p$ is a precompact metrizable topology.
- Any product of topological groups endowed with the product topology.
The dual of a topological group

All the groups in the sequel will be abelian. We omit this term.

- Dualizing object: the topological group $\mathbb{T}$.
- Homomorphisms from any group $G$ to $\mathbb{T}$ are called characters. The set of all characters $G^a := \text{Hom}(G, \mathbb{T})$ has a group structure with respect to the pointwise operation.
- For a topological group $(G, \tau)$, the continuous characters form a subgroup of $G^a$ called the dual group of $(G, \tau)$. It will be denoted by $G^\wedge := \text{CHom}(G, \mathbb{T}) \leq \text{Hom}(G, \mathbb{T})$.
- $\mathbb{T}_+ := \{x \in \mathbb{T}, \text{Re}x \geq 0\}$

### Duals of some elementary groups

1. $\mathbb{Z}^\wedge \cong \mathbb{T}$.
2. $\mathbb{T}^\wedge \cong \mathbb{Z}$.
3. $\mathbb{R}^\wedge \cong \mathbb{R}$.
4. For $n > 0$, $(\mathbb{R}^n)^\wedge \cong \mathbb{R}^n$. 

Mackey-Arens Theorem for Abelian topological groups.
Reflexive groups

The Pontryagin dual of a topological group \((G, \tau)\) is the pair \((G^\wedge, \tau_{co})\), where \(\tau_{co}\) denotes the compact-open topology. Briefly: \(G^\wedge_{co} : = (G^\wedge, \tau_{co})\), and \(G^{\wedge\wedge} : = (G^\wedge_{co})^\wedge_{co}\) is the bidual.

**Definition**

A group \((G, \tau)\) is reflexive if the canonical mapping:

\[
\alpha_G : G \longrightarrow G^{\wedge\wedge}
\]

\[
g \longmapsto \alpha_G(g) : G^\wedge \longrightarrow \mathbb{T}
\]

\[
\gamma \longmapsto \alpha_G(\gamma) : = \gamma(g)
\]

is a topological isomorphism between \(G\) and its bidual \(G^{\wedge\wedge}\).

**Pontryagin Duality Theorem (1934-1935)**

Every locally compact abelian group (LCA- group) is reflexive.

In particular: \(\mathbb{R}, \mathbb{T}, \mathbb{Z}, \mathbb{R}^n\) are reflexive groups. A Banach space is a reflexive group (M. Smith, 1951).
Duality for abelian groups

Let $G$ be a group and $F \leq G^a := \text{Hom}(G, \mathbb{T})$. The pair $(G, F)$ is a group duality. If $(G, \tau)$ is a $tg$, there is a natural duality: $(G, G^\wedge)$.

Topologies associated to the group duality $(G, F)$:

- $\sigma(G, F)$ the weak topology on $G$ corresponding to the elements of $F$. It is a precompact group topology. It is Hausdorff whenever $F$ separates the points of $G$.

- $\sigma(F, G)$ the topology on $F$ of pointwise convergence on the elements of $G$. It is a Hausdorff precompact topology.
Vilenkin’s contribution

Quasi-convex subsets of a tg.

Let \((G, \tau)\) be a tg. A subset \(S \subseteq G\) is **quasi-convex** if for any \(x \in G \setminus S\) there exists \(\chi \in G^\wedge\) such that \(\chi(S) \subset T_+\) and \(\chi(x) \notin T_+\).

**Example:** \([-1, 0, 1] \subset \mathbb{R}\) is quasi-convex in \(\mathbb{R}_u\). Obviously non convex!!!.

Properties of quasi-convex subset

Let \((G, \tau)\) be a tg and \(S \subset G\) quasi-convex. The following hold:

- \(0 \in S\) and \(S\) is symmetric.
- \(S\) is closed, since \(S = \bigcap_{\chi \in S} \chi^{-1}(T_+)\)

**Definition**

A tg \(G\) is **locally quasi-convex** (lqc) if it has a basis of 0-nbhs that are quasi-convex.
**Definition**

Let $\mathcal{T} = (G, \tau)$ be a tg, $S \subseteq G$ and $A \subseteq G\wedge$.

- **The polar of $S$ is:** $S^\triangledown := \{ \chi \in G\wedge \mid \chi(S) \subseteq \mathbb{T}_+ \}$.

- **The prepolar of $A$ is:**
  
  $A^\triangledown := \{ x \in G \mid \chi(x) \in \mathbb{T}_+ \text{ for all } \chi \in A \}$.

The following statements hold:

- $S^\triangledown$ is quasi-convex in $(G\wedge, \sigma(G\wedge, G))$ and $A^\triangledown$ is quasi-convex in $(G, \tau)$.

- The family $\mathcal{U} := \{ K^\triangledown \mid K \subseteq G \text{ compact} \}$ is a basis of 0-nbhs for the compact-open topology in $G\wedge$.

- A subset $L \subseteq G\wedge$ is equicontinuous (with respect to $\tau$) if there exists a 0-nbhd $U \subset G$, such that $L \subseteq U^\triangledown$.

- The polar $V^\triangledown$ of a 0-nbhd $V \subset G$, is equicontinuous and $\sigma(G\wedge, G)$-compact.
Consequences easily derived:

- The dual group of a discrete group is compact
- The dual group of a compact group is discrete.
- The dual group of an LCA-group is LCA.
Examples and stability properties

- Any dual group, say $G^\wedge$, is lqc.
- Reflexive groups are lqc. In particular, LCA-groups are lqc.
- Any subgroup of a lqc-group is lqc.
- Arbitrary products of lqc-groups are lqc-groups.
- Hausdorff quotients of lqc-groups are not in general lqc.
- If $(G, \tau)$ has sufficiently many continuous characters, $(G, \sigma(G, G^\wedge))$ is lqc.
- Every locally convex space is a lqc-group.
Auxiliary notions for the definition of the Mackey topology for a topological group:

**Auxiliary definitions**

- Let $(G, \tau)$ be a tg. A group topology $\nu$ on $G$ is **compatible** with $\tau$ if $(G, \tau)^{\wedge} = (G, \nu)^{\wedge}$ (equality as sets).
- $\mathcal{LQC}(G_{\tau})$ denotes the family of all locally quasi-convex topologies on $G$ that are compatible with $\tau$.

**Mackey topology and Mackey group**

- Let $(G, \tau)$ be a tg. A lqc topology $\mu$ on $G$ is **the Mackey topology** for $(G, \tau)$ if it is the top element in $\mathcal{LQC}(G_{\tau})$.
- If $(G, \tau)$ carries the Mackey topology (i.e. $\tau = \mu$) then $(G, \tau)$ is a **Mackey group**.
Mackey and non-Mackey groups

**Theorem (Chasco, M-P, Tarieladze (1999))**

*Every lqc Hausdorff and completely metrizable is a Mackey group.*

**A metrizable non complete Mackey group**

\[ \mathbb{Z}_p^{(\mathbb{N})} \] (direct sum of countably many copies of the cyclic group \( \mathbb{Z}_p \)), with the topology induced by the product in \( \mathbb{Z}_p^{\mathbb{N}} \).

A metrizable lqc group may not be Mackey:

**Dikranjan, M-P, Tarieladze (2010)**

Let \( X \neq \{0\} \) be a connected, compact metrizable group. the group \( c_0(X) \subset X^{\mathbb{N}} \) of the null sequences on \( X \), with the topology induced by the product in \( X^{\mathbb{N}} \), is not a Mackey group.
Other non-Mackey groups:

**Aussenhofer, de la Barrera (2011)**

A linear, nondiscrete topology on $\mathbb{Z}$ is not a Mackey topology. Therefore, $(\mathbb{Z}, \tau_p)$ (the integers with the $p$-adic topology) is not a Mackey group.


The group of rationals $\mathbb{Q}_u$ is not a Mackey group.

**Open questions.**

- Has $\mathbb{Q}_u$ a Mackey topology?.
- Is there a Mackey topology in the duality $(\mathbb{Z}, \mathbb{Z}(p^{\infty}))$?
The failure of Mackey-Arens Theorem for abelian tg

Free abelian topological group over a Tychonoff space.

For a Tychonoff space $X$, there exists a topological abelian group $A_G(X)$, unique with the following characteristics:

- The supporting set of $A_G(X)$ is the free abelian group over $X$.
- $X$ is topologically embedded in $A_G(X)$ as a closed subset.
- If $Y$ is an abelian topological group and $f : X \to Y$ a continuous mapping, $f$ can be uniquely “extended” to a continuous homomorphism from $A_G(X)$ into $Y$.

$A_G(X)$ is called the free abelian topological group over $X$. Its existence was proved by Graev.

Aussenhofer, Gabriyelyan (2018)

The free abelian topological group $A_G(s)$ where $s$ is the null sequence $s := \{0\} \cup \{1/n : n \in \mathbb{N}\}$ does not have a Mackey topology.
**g-barrelled groups**

**Definition (Chasco, M - P , Tarieladze 1999)**

A topological group \((G, \tau)\) is g-barrelled if every \(\sigma(G^\wedge, G)\)-compact \(L \subseteq G^\wedge\) is equicontinuous.

**Theorem (Chasco, M - P , Tarieladze 1999)**

Every g-barrelled and lqc group \((G, \tau)\) is a Mackey group. Furthermore, \(\tau\) is the topology of uniform convergence on the \(\sigma(G^\wedge, G)\)-compact subsets of \(G^\wedge\).

**Corollary**

For a \(t\)g \((G, \tau)\) there is at most one g-barrelled topology in \(\mathcal{LQC}(G_\tau)\).

It may happen that there are many g-barrelled compatible topologies, non locally quasi-convex.
Classes of $g$-barrelled groups

- Metrizables, hereditariamente Baire [Chasco, M.P., Tarieladze, 1999].
- Baire separable [idem, 1999].
- Čech completos (in particular, todo grupo localmente compacto abeliano) [idem, 1999].
- Pseudocompact groups [Hernández, Macario, 2003].
- Precompact, Baire, with bounded torsion [Chasco, Domínguez, Tkachenko, 2017].
Non \( g \)-barrelled topological groups

- \( \mathbb{Q}u \).
- Any countable MAP group.

The dual group of a \( g \)-barrelled group may not be \( g \)-barrelled:

\[
c_0(T) := \{(t_n)_{n \in \mathbb{N}} \in T^\mathbb{N} : \lim_{n} |t_n - 1| = 0 \}.
\]

with the metric \( d : c_0(T) \times c_0(T) \rightarrow \mathbb{R}_+ \) defined by:

\[
d(s, t) = \max_{n \in \mathbb{N}} |s_n - t_n|, \quad t = (t_n)_{n \in \mathbb{N}} \in c_0(T), \quad s = (s_n)_{n \in \mathbb{N}} \in c_0(T).
\]

\((c_0(T), d)\) is a complete metric space. Its dual is isomorphic to the direct sum of countably many copies of \( \mathbb{Z} \), say \( \mathbb{Z}^{(\mathbb{N})} \). 

\( c_0(T) \) es \( g \)-barrelled, pero su dual \((c_0(T))^\wedge\) no lo es.
The products, the direct sums and the inductive limits of \( g \)-barrelled groups are \( g \)-barrelled.

**Definition (Frolik, 1961)**

A topológical space \((X, \tau)\) is strongly countably complete \( \text{SCC} \) if there exists a sequence of open coverings \( \{A_n, n \in \mathbb{N}\} \) such that any decreasing sequence \( \mathcal{F} \) of closed nonempty subsets of \( X \), has nonempty intersection, provided that \( \mathcal{F} \) contains \( A_i \)-small sets for every \( i \in \mathbb{N} \).
Definition

A regular topological space $X$ is a Namioka space if for all compact space $Y$, for all metrizable space $Z$ and for all $f : Y \times X \to Z$ separately continuous, there exists a $G_\delta$-dense subset $A \subset X$, such that $f$ is jointly continuous for every point in $Y \times A$.

Theorem (Namioka)

Every regular SCC topological space is a Namioka space.

Domínguez, M-P, Tarieladze

Namioka topological groups are $g$-barrelled. Therefore SCC-groups are also $g$-barrelled.
The conditions 1) and 2) of the Mackey-Arens Theorem may be non-equivalent for topological groups, as the following example shows:

(Bonales, Trigos-Arrieta, Vera Mendoza, 2003)

Let $G := \mathbb{Z}_5^{(\mathbb{N})}$ (direct sum of countably many copies of the cyclic group of order 5), with the topology $\nu$ induced by the product in $\mathbb{Z}_5^\mathbb{N}$. The following holds:

- It is lqc and $|\mathcal{LQC}(G_\tau)| = 1$. Therefore $\nu$ is the Mackey topology.
- The topology on $G$ of uniform convergence on the $\sigma(G^\wedge, G)$-compact subsets of $G^\wedge$ is discrete, thus different of the original topology $\nu$.
- $(G, \nu)$ is not $g$-barrelled.

The above group $G$ is Mackey but non $g$-barrelled.


