Twisted sums of $c_0$ and $C(K)$: Scattered Spaces

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Question

Assume $K$ is a nonmetrizable scattered compact space. Is there a nontrivial twisted sum of $c_0$ and $C(K)$?
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**Definition**

A topological space is said to be scattered if every nonempty subspace has an isolated point with respect to the subspace topology.
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Example

If $\Gamma$ is a discrete topological space, then its one-point compactification $\Gamma \cup \{\infty\}$ is scattered.
Definition

Let $\mathcal{X}$ be a topological space. We define by recursion on $\alpha$ a decreasing family of closed subsets of $\mathcal{X}$:

- $\mathcal{X}^{(0)} = \mathcal{X}$;
- For every ordinal $\alpha$, $\mathcal{X}^{(\alpha+1)} = \mathcal{X}^{(\alpha)} \setminus \text{Is}(\mathcal{X}^{(\alpha)})$, where $\text{Is}(\mathcal{X}^{(\alpha)})$ denotes the set of isolated points of $\mathcal{X}^{(\alpha)}$;
- For every limit ordinal $\alpha$, $\mathcal{X}^{(\alpha)} = \bigcap_{\beta \in \alpha} \mathcal{X}^{(\beta)}$.

The space $\mathcal{X}^{(\alpha)}$ is called the $\alpha^{th}$ Cantor–Bendixson derivative of $\mathcal{X}$.
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Proposition

A topological space $\mathcal{X}$ is scattered if and only if there exists an ordinal $\alpha$ such that $\mathcal{X}^{(\alpha)} = \emptyset$. 
Definition

If $\mathcal{X}$ is scattered, then the height of $\mathcal{X}$ is defined as the least ordinal $\alpha$ such that $\mathcal{X}^{(\alpha)} = \emptyset$. If the height of $\mathcal{X}$ is a natural number, then we say that $\mathcal{X}$ has finite height.
**Definition**

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**Theorem (Castillo, Top. Appl.–2016)**

Assume CH. If $K$ is a nonmetrizable finite height compact space, then there exists a nontrivial twisted sum of $c_0$ and $C(K)$.
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Theorem (Castillo, Top. Appl.–2016)

Assume CH. If $K$ is a nonmetrizable finite height compact space, then there exists a nontrivial twisted sum of $c_0$ and $C(K)$.

The proof of this result was obtained using homological tools.

Question

Does it hold in ZFC that if $K$ is a nonmetrizable finite height compact space, then there exists a nontrivial twisted sum of $c_0$ and $C(K)$?

Answer: No.
### Definition

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### Question

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Definition

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Theorem (Castillo, Top. Appl.–2016)

Assume CH. If $K$ is a nonmetrizable finite height compact space, then there exists a nontrivial twisted sum of $\ell^1$ and $C(K)$.

The proof of this result was obtained using homological tools.

Question

Does it hold in ZFC that if $K$ is a nonmetrizable finite height compact space, then there exists a nontrivial twisted sum of $\ell^1$ and $C(K)$?

Answer: No.
Theorem (Marciszewski and Plebanek, JFA–2018)

Assume $\text{MA} + \neg \text{CH}$. If $K$ is a separable compact space with height 3 and $w(K) < c$, then every twisted sum of $c_0$ and $C(K)$ is trivial.
Theorem (Marciszewski and Plebanek, JFA–2018)

Assume $\text{MA} \vdash \neg \text{CH}$. If $K$ is a separable compact space with height 3 and $w(K) < c$, then every twisted sum of $c_0$ and $C(K)$ is trivial.

Idea of the proof:

- Assume $\text{MA} \vdash \neg \text{CH}$. Let $K$ be a boolean separable space with $w(K) < c$. If the boolean algebra of clopen subsets of $K$ has the local extension property, then every twisted sum of $c_0$ and $C(K)$ is trivial.
Theorem (Marciszewski and Plebanek, JFA–2018)

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Idea of the proof:

- Assume $MA + \neg CH$. Let $K$ be a boolean separable space with $w(K) < c$. If the boolean algebra of clopen subsets of $K$ has the local extension property, then every twisted sum of $c_0$ and $C(K)$ is trivial.
- if $K$ has height 3, then $Clop(K)$ has the local extension property.
Theorem (Correa and Tausk, Fund. Math.–2019)
If \( K \) is a compact space with finite height, then \( \text{Clop}(K) \) has the local extension property.

Corollary
Assume MA + \( \neg \) CH. If \( K \) is a separable compact space with finite height and \( \omega(K) < c \), then every twisted sum of \( c_0 \) and \( \text{C}(K) \) is trivial.

Corollary
Let \( K \) be a separable finite height compact space with \( \omega(K) < c \). Then the existence of nontrivial twisted sums of \( c_0 \) and \( \text{C}(K) \) is independent of the axioms of ZFC.
Theorem (Correa and Tausk, Fund. Math.–2019)

If $K$ is a compact space with finite height, then $\text{Clop}(K)$ has the local extension property.

Corollary

Assume $\text{MA}+\neg\text{CH}$. If $K$ is a separable compact space with finite height and $w(K) < c$, then every twisted sum of $c_0$ and $C(K)$ is trivial.

Corollary

Let $K$ be a separable finite height compact space with $w(K) < c$. Then the existence of nontrivial twisted sums of $c_0$ and $C(K)$ is independent of the axioms of ZFC.
Question

What happens, under $\text{MA}+\neg\text{CH}$, if $K$ is a nonseparable finite height compact space with $w(K) < c$?
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What happens, under $\text{MA} \vdash \lnot \text{CH}$, if $K$ is a nonseparable finite height compact space with $w(K) < c$?

Theorem (Correa, Fund. Math.–2019)

Assume $\text{MA} \vdash \lnot \text{CH}$. If $K$ is a nonseparable scattered compact space with $w(K) < c$, then there exists a nontrivial twisted sum of $c_0$ and $C(K)$. 
Let $\gamma \omega$ be a compactification of $\omega$ and $\gamma \omega \setminus \omega$ be the reminder of this compactification. If $R : C(\gamma \omega) \to C(\gamma \omega \setminus \omega)$ denotes the restriction operator, then $\text{Ker}(R)$ is isomorphic to $c_0$ and therefore the following is a twisted sum of $c_0$ and $C(\gamma \omega \setminus \omega)$:

$$0 \to c_0 \to C(\gamma \omega) \xrightarrow{R} C(\gamma \omega \setminus \omega) \to 0.$$
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(van Douwen and Przymusiński-1980) Assume $MA + \neg CH$. If $K$ is a compact Hausdorff space with $w(K) < c$, then $K$ is homeomorphic to the reminder of a compactification of $\omega$. 

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(W. Kubiś) If the above twisted sum is trivial, then $\gamma \omega \setminus \omega$ is the support of a strictly positive regular measure.

(van Douwen and Przymusiński-1980) Assume $\text{MA} + \neg \text{CH}$. If $K$ is a compact Hausdorff space with $\omega(K) < \mathfrak{c}$, then $K$ is homeomorphic to the reminder of a compactification of $\omega$.

If $K$ is a compact and scattered space, then every regular measure has separable support.
Question

What happens, under MA+$\neg$ CH, if $K$ is a nonseparable compact scattered space with $w(K) = c$?
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What happens, under MA + ¬ CH, if K is a nonseparable compact scattered space with \( w(K) = c \)?

Assume MA + ¬ CH. If K is a compact Hausdorff space with \( w(K) < c \), then K is homeomorphic to the reminder of a compactification of \( \omega \).
Question

What happens, under MA + ¬ CH, if $K$ is a nonseparable compact scattered space with $w(K) = c$?


Assume MA + ¬ CH. If $K$ is a compact Hausdorff space with $w(K) < c$, then $K$ is homeomorphic to the reminder of a compactification of $\omega$.

Question

Does it hold under MA + ¬ CH that every compact Hausdorff space $K$ with $w(K) = c$ is homeomorphic to the reminder of a compactification of $\omega$?
What happens, under $\text{MA} + \neg \text{CH}$, if $K$ is a nonseparable compact scattered space with $w(K) = \mathfrak{c}$?


Assume $\text{MA} + \neg \text{CH}$. If $K$ is a compact Hausdorff space with $w(K) < \mathfrak{c}$, then $K$ is homeomorphic to the reminder of a compactification of $\omega$.

Does it hold under $\text{MA} + \neg \text{CH}$ that every compact Hausdorff space $K$ with $w(K) = \mathfrak{c}$ is homeomorphic to the reminder of a compactification of $\omega$?

**Answer:** No.
Theorem (Frankiewicz, Fund. Math.–1985)

*It is relatively consistent with MA + ¬CH the existence of a compact Hausdorff space with weight ℵ that is not homeomorphic to the reminder of a compactification of ω. This space is not scattered.*
Theorem (Frankiewicz, Fund. Math.–1985)

It is relatively consistent with $\text{MA} + \neg \text{CH}$ the existence of a compact Hausdorff space with weight $\mathfrak{c}$ that is not homeomorphic to the reminder of a compactification of $\omega$. This space is not scattered.

Open Problem

Does it hold under $\text{MA} + \neg \text{CH}$ that every compact scattered space $K$ with $w(K) = \mathfrak{c}$ is homeomorphic to the reminder of a compactification of $\omega$?
Assume $\text{MA} + \neg \text{CH}$. Let $K$ be a compact Hausdorff and scattered space with $w(K) = c$. If

$$\left| \{ p \in K : w(K, p) > \omega \} \right| < c,$$

then $K$ is homeomorphic to the reminder of a compactification of $\omega$.

$$w(K, p) = \min \{ w(V) : V \text{ is a nhood of } p \}$$
Theorem (Correa, Fund. Math.–2019)

Assume MA + ¬CH. Let K be a compact Hausdorff and scattered space with \( w(K) = c \). If

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\left| \{ p \in K : w(K, p) > \omega \} \right| < c,
\]

then K is homeomorphic to the remainder of a compactification of \( \omega \).

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w(K, p) = \min\{ w(V) : V \text{ is a nhood of } p \}\]
Question

Does it hold in ZFC that if $K$ is a finite height compact space with $w(K) \geq c$, then there exists a nontrivial twisted sum of $c_0$ and $C(K)$?
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Does it hold in ZFC that if $K$ is a finite height compact space with $w(K) \geq \mathfrak{c}$, then there exists a nontrivial twisted sum of $c_0$ and $C(K)$?

Answer: Yes.

Theorem (Avilés, Marciszewski and Plebanek–2019)

If $K$ is a finite height compact space with $w(K) \geq \mathfrak{c}$, then there exists a nontrivial twisted sum of $c_0$ and $C(K)$. 
Thanks for your attention!
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Nonseparable $c(k)$-spaces can be twisted when $k$ is a finite
height compact.

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Nontrivial twisted sums for finite height spaces under martin’s
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Local extension property for finite height spaces.
